Stochastic Algorithms for Solving Nonlinearly Constrained Continuous Optimization Problems

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involving joint work with

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Outline

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Mathematical optimization

Main area of research: **mathematical optimization**; design and analysis of algorithms to solve

$$\min_{x \in \mathcal{X}} \; f(x) \; \text{ s.t. } \; c_{\mathcal{E}}(x) = 0 \text{ and } c_{\mathcal{I}}(x) \leq 0.$$

Nonconvex continuous optimization, where f, $c_{\mathcal{E}}$, and $c_{\mathcal{I}}$ are continuous and not necessarily convex.

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Disclaimer: Research not on *solving* (to "global optimality") these problems.

- Prior to 10 years ago: Exact derivatives, research to overcome expense of subproblems
- Last 10 years: Stochastic or noisy derivative estimates, data-driven optimization

Motivation and challenges: Stochastic algorithms for constrained optimization

Primarily motivated by informed learning problems where regularization is ineffective or inefficient; e.g.,

- physics-informed machine learning
- fair (supervised) machine learning
- . . . but algorithms are general-purpose, applicable for other network/simulation optimization problems

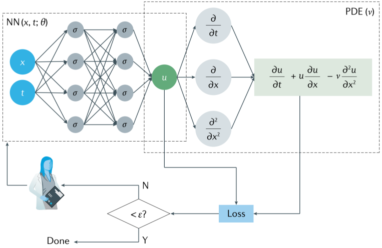

Photo: Karniadakis et al.

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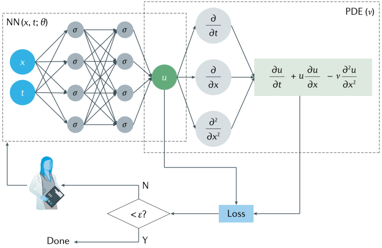
Same challenges and questions as for unconstrained:

- convergence/complexity guarantees (adaptive algorithms)
- computational complexity
- stability guarantees
- generalization properties



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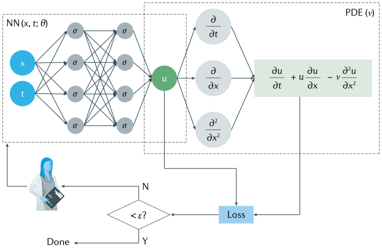
New challenges for *handling constraints as constraints*:

- (i.e., avoid penalty methods, augmented Lagrangian, etc.)
- balancing the objective and constraints
- degeneracy and infeasibility



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Learning: Prediction function

Our aim is to determine a prediction function $p \in \mathcal{P}$, where $\mathcal{P}$ is some family of functions, such that

$$p(a_j)$$

yields an accurate prediction corresponding to any given input feature vector a_j.

Learning: Prediction function, parameterized

Let us say that the family is parameterized by some vector x such that

$$p(a_j, x)$$

yields an accurate prediction corresponding to any given input feature vector a_j.

Learning: Supervised

In the context of supervised learning, we have known input-output pairs $\{(a_j, b_j)\}_{j=1}^{n_o}$, then

$$\min_{x \in \mathbb{R}^n} \frac{1}{n_o} \sum_{j=1}^{n_o} \ell(p(a_j, x), b_j)$$

becomes our empirical-loss training problem to determine the optimal parameter vector x.

Learning: Supervised and regularized

If, in addition, we aim to impose some structure on the solution x, then we may consider

$$\min_{x \in \mathbb{R}^n} \frac{1}{n_o} \sum_{j=1}^{n_o} \ell(p(a_j, x), b_j) + r(x)$$

where r is a *regularization* function.

Learning: Supervised and regularized

If, in addition, we aim to impose some structure on the solution x, then we may consider

$$\min_{x \in \mathbb{R}^n} \frac{1}{n_o} \sum_{j=1}^{n_o} \ell(p(a_j, x), b_j) + r(x)$$

where r is a *regularization* function. But is this the right approach for *informed* learning?

Learning: Supervised and informed with *soft* constraints

Added to the loss (e.g., mean-squared error or other data-fitting term), we might consider

$$\min_{x \in \mathbb{R}^n} \; \frac{1}{n_o} \sum_{j=1}^{n_o} \ell(p(a_j, x), b_j) + \frac{1}{n_c} \sum_{j=1}^{n_c} \phi(p(\tilde{a}_j, x), \dots, \tilde{b}_j)$$

where $\{(\tilde{a}_j, \tilde{b}_j)\}_{j=1}^{n_c}$ are some known input-output pairs and ϕ encodes known information.

Learning: Supervised and informed through layer design

Another viable approach is to embed information through the prediction function itself such that

$$\min_{x\in\mathbb{R}^n} \frac{1}{n_o}\sum_{j=1}^{n_o} \ell(\hat{p}(a_j, x), b_j)$$

ensures that information is enforced with every forward pass. (Expense?)

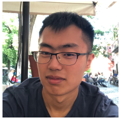
Learning: Supervised and informed with *hard* constraints

Back to the "original" family for p, how about imposing hard constraints during training, as in

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \ & \frac{1}{n_o}\sum_{j=1}^{n_o} \ell(p(a_j,x),b_j) \\ \text{s.t.} \ & \varphi(p(\tilde{a}_j,x),\dots,\tilde{b}_j) = 0 \ (\text{or} \leq 0) \ \text{ for all } \ i \in \{1,\dots,n_c\} \end{aligned}$$

such that we restrict attention to functions that are informed implicitly?

Collaborators and references

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Expected-loss training problems

For the sake of generality/generalizability, the expected-loss objective function is

$$\int_{\mathcal{A}\times\mathcal{B}} \ell(p(a,x),b)\mathrm{d}\mathbb{P}(a,b) \equiv \mathbb{E}_\omega[F(x,\omega)] =: f(x).$$

One might consider various paradigms for imposing the constraints:

- expectation constraints
- (distributionally) robust constraints
- probabilistic (i.e., chance) constraints

For our recent work, we consider constraints whose values and derivatives can be computed:

$$c_{\mathcal{E}}(x) = 0 \quad \text{and} \quad c_{\mathcal{I}}(x) \leq 0$$

e.g., as in imposing a fixed set of constraints corresponding to a fixed set of sample data.

Outline

Stochastic gradient method

Consider $\min_{x \in \mathbb{R}^n} f(x)$, where $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with constant L.

Algorithm SG : Stochastic gradient method

1: choose an initial point $x_1 \in \mathbb{R}^n$ and step sizes $\{\alpha_k\} > 0$
2: **for** $k \in \{1, 2, \dots\} =: \mathbb{N}$ **do**
3: set $x_{k+1} \leftarrow x_k - \alpha_k g_k$, where $g_k \approx \nabla f(x_k)$
4: **end for**

Algorithm† behavior is defined by $(\Omega, \mathcal{F}, \mathbb{P})$, where

- $\Omega = \Gamma \times \Gamma \times \Gamma \times \cdots$ (sequence of draws determining stochastic gradients);
- $\mathcal{F}$ is a σ-algebra on Ω, specifically, the set of events (i.e., measurable subsets of Ω); and
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure.

One can view any $\{(x_k, g_k)\}$ as a realization of $\{(X_k, G_k)\}$, where for all $k \in \mathbb{N}$

$$x_k = X_k(\omega) \text{ and } g_k = G_k(\omega) \text{ given } \omega \in \Omega.$$

† Robbins and Monro (1951); Sutton Monro = former Lehigh ISE faculty member

Convergence of SG

Let $\mathbb{E}[\cdot]$ denote expectation with respect to $\mathbb{P}[\cdot]$. Analyze through associated sub-σ-algebras $\{\mathcal{F}_k\}$.

Assumption

For all $k \in \mathbb{N}$, one has that

- $\mathbb{E}[G_k|\mathcal{F}_k] = \nabla f(X_k)$ *and*
- $\mathbb{E}[\|G_k\|_2^2|\mathcal{F}_k] \leq M + M_{\nabla f}\|\nabla f(X_k)\|_2^2$

By Lipschitz continuity of ∇f and construction of the algorithm, one finds

$$\begin{aligned} f(X_{k+1}) - f(X_k) &\leq \nabla f(X_k)^T(X_{k+1} - X_k) + \tfrac{1}{2}L\|X_{k+1} - X_k\|_2^2 \\ &= -\alpha_k \nabla f(X_k)^T G_k + \tfrac{1}{2}\alpha_k^2 L\|G_k\|_2^2 \\ \implies \mathbb{E}[f(X_{k+1})|\mathcal{F}_k] - f(X_k) &\leq -\alpha_k\|\nabla f(X_k)\|_2^2 + \tfrac{1}{2}\alpha_k^2 L\mathbb{E}[\|G_k\|_2^2|\mathcal{F}_k] \\ &\leq -\alpha_k\|\nabla f(X_k)\|_2^2 + \tfrac{1}{2}\alpha_k^2 L(M + M_{\nabla f}\|\nabla f(X_k)\|_2^2), \end{aligned}$$

where the last inequalities follow by the assumption and since $f(X_k)$ and $\nabla f(X_k)$ are $\mathcal{F}_k$-measurable.

SG theory

Taking total expectation, one arrives at

$$\mathbb{E}[f(X_{k+1}) - f(X_k)] \leq -\alpha_k(1 - \tfrac{1}{2}\alpha_k L M_{\nabla f})\mathbb{E}[\|\nabla f(X_k)\|_2^2] + \tfrac{1}{2}\alpha_k^2 LM$$

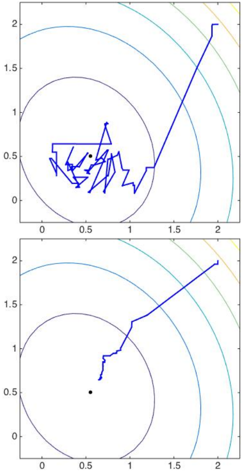
Theorem

$$\alpha_k = \frac{1}{LM_{\nabla f}} \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k}\|\nabla f(X_j)\|_2^2\right] \leq M_k \xrightarrow{k\to\infty} \mathcal{O}\left(\frac{M}{M_{\nabla f}}\right)$$

$$\alpha_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\alpha_j\right)}\sum_{j=1}^{k}\alpha_j\|\nabla f(X_j)\|_2^2\right] \to 0$$

$$\implies \liminf_{k\to\infty}\ \mathbb{E}[\|\nabla f(X_k)\|_2^2] = 0$$

(further steps) $\implies \nabla f(X_k) \to \infty$ *almost surely.*



Constrained optimization

Consider

$$\boxed{\begin{aligned} \min_{x\in\mathbb{R}^n}\ & f(x) \\ \text{s.t. } & c(x) = 0 \end{aligned}}$$

Option: Regularization / soft constraints (penalization), as in

$$\min_{x\in\mathbb{R}^n}\ \tau f(x) + \|c(x)\|_q^p \ (+y^T c(x)),$$

then employ a (stochastic) algorithm for unconstrained optimization.

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On the positive side, "exact" penalty function theory has been well established for decades:

- *can* solve the constrained problem, in theory.

Unfortunately, however, such an approach is not ideal:

- appropriate balance (τ and/or y) not known in advance
- $p = 1$ (nonsmooth), $p = 2$ (need $\tau \searrow 0$, ill-conditioning)

Sequential quadratic optimization (SQP)

Consider

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t. } c(x) = 0$$

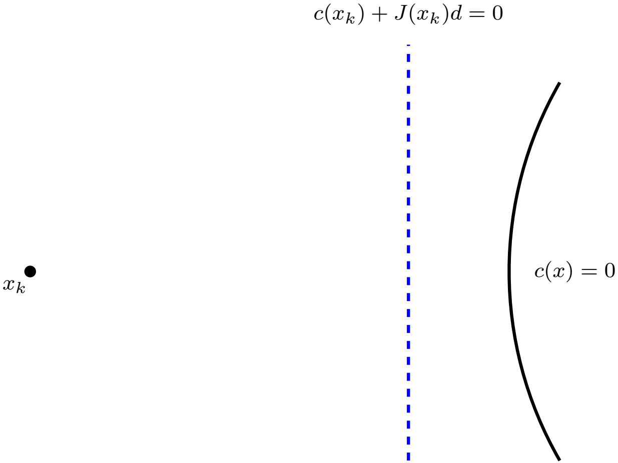
Option: With $J \equiv \nabla c^T$ and H positive definite over Null(J), two viewpoints:

$$\begin{bmatrix}\nabla f(x) + J(x)^T y \\ c(x)\end{bmatrix} = 0 \quad \text{or} \quad \min_{d\in\mathbb{R}^n} f(x) + \nabla f(x)^T d + \tfrac{1}{2} d^T H d \quad \text{s.t. } c(x) + J(x)d = 0$$

both leading to the same "Newton-SQP system":

$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} d_k \\ y_k \end{bmatrix} = - \begin{bmatrix} \nabla f(x_k) \\ c_k \end{bmatrix}$$

SQP illustration



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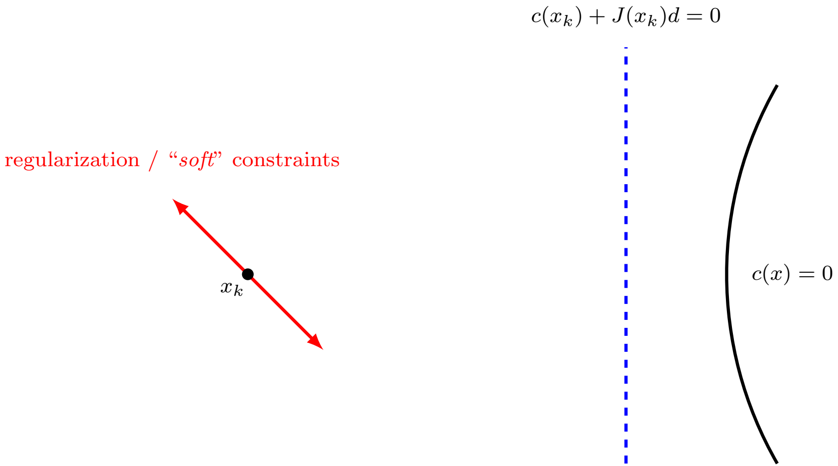

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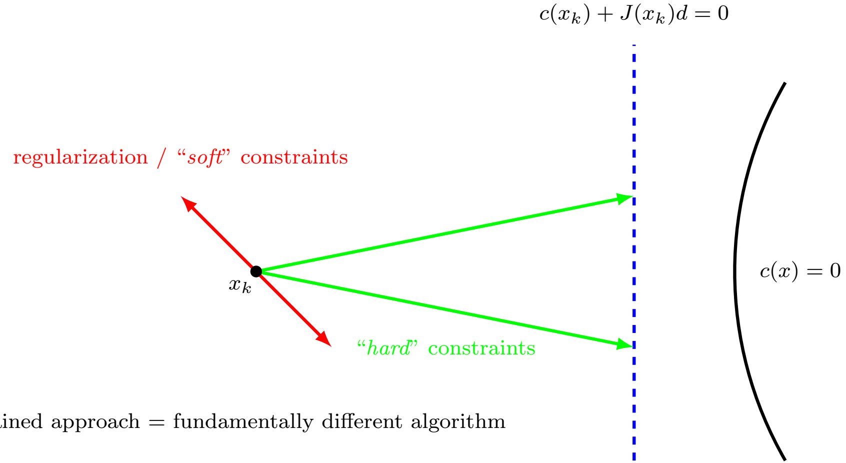
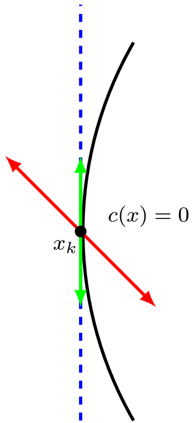


constrained approach = fundamentally different algorithm

SQP illustration

regularization / "*soft*" constraints

"*hard*" constraints $\implies$ step in null space



Stochastic SQP

Algorithm guided by merit function with adaptive parameter τ defined by

$$\phi(x,\tau) = \tau f(x) + \|c(x)\|_1$$

Algorithm : Stochastic SQP

choose $x_1 \in \mathbb{R}^n$, $\tau_0 \in (0,\infty)$, $\{\beta_k\} \in (0,1]^{\mathbb{N}}$
for $k \in \{1,2,\ldots\}$ **do**
    compute step: solve

$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} d_k \\ y_k \end{bmatrix} = -\begin{bmatrix} g_k \\ c_k \end{bmatrix}$$

    update merit parameter: set τ_k to ensure

$$\phi'(x_k,\tau_k,d_k) \leq -\Delta q(x_k,\tau_k,g_k,d_k) \ll 0$$

    compute step size: set

$$\alpha_k = \Theta\left(\frac{\beta_k \tau_k}{\tau_k L_{\nabla f} + L_{\nabla c}}\right)$$

    then $x_{k+1} \leftarrow x_k + \alpha_k d_k$
end for

Convergence theory in *deterministic setting*

Assumption

- f, c, ∇f, *and* J *bounded and Lipschitz*
- *singular values of* J *bounded below (i.e., the LICQ)*
- $u^T H_k u \geq \zeta \|u\|_2^2$ *for all* $u \in \mathrm{Null}(J_k)$ *for all* $k \in \mathbb{N}$

Theorem

- $\{\alpha_k\} \geq \alpha_{\min}$ *for some* $\alpha_{\min} > 0$
- $\{\tau_k\} \geq \tau_{\min}$ *for some* $\tau_{\min} > 0$
- $\Delta q(x_k, \tau_k, \nabla f(x_k), d_k) \to 0$ *implies optimality error vanishes, specifically,*

$$\|d_k\|_2 \to 0, \quad \|c_k\|_2 \to 0, \quad \|\nabla f(x_k) + J_k^T y_k\|_2 \to 0$$

Stochastic setting: What do we want?

What we want/expect from the algorithm?

Note: We are interested in the stochastic approximation (SA) regime.

Ultimately, there are *many* questions to answer:

- convergence guarantees
- complexity guarantees
- tradeoff analysis (Bottou and Bousquet)
- generalization
- large-scale implementations
- beyond first-order (SG) methods

Fundamental lemma

Recall in the unconstrained setting that

$$\mathbb{E}[f(X_{k+1})|\mathcal{F}_k] - f(X_k) \leq -\alpha_k \|\nabla f(X_k)\|_2^2 + \frac{1}{2}\alpha_k^2 L \mathbb{E}[\|G_k\|_2^2|\mathcal{F}_k]$$

Lemma

For all $k \in \mathbb{N}$ one finds (before taking expectations)

$$\begin{aligned} &\phi(X_{k+1}, \mathcal{T}_{k+1}) - \phi(X_k, \mathcal{T}_k) \\ &\leq \underbrace{-\mathcal{A}_k \Delta q(X_k, \mathcal{T}_k, \nabla f(X_k), D_k^{\text{true}})}_{\mathcal{O}(\beta_k),\ \text{“deterministic”}} + \underbrace{\tfrac{1}{2}\mathcal{A}_k \beta_k \Delta q(X_k, \mathcal{T}_k, G_k, D_k)}_{\mathcal{O}(\beta_k^2),\ \text{stochastic/noise}} + \underbrace{\mathcal{A}_k \mathcal{T}_k \nabla f(X_k)^T (D_k - D_k^{\text{true}})}_{\text{due to adaptive } \mathcal{A}_k} \end{aligned}$$

Good merit parameter behavior

Theorem 4

Let $\mathcal{E} :=$ *event that* $\{\mathcal{T}_k\}$ *eventually remains constant at* $\mathcal{T}' \geq \tau_{\min} > 0$. *Then, conditioned on* $\mathcal{E}$,

$$\beta_k = \Theta(1) \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k}\Delta q(X_j, \mathcal{T}', \nabla f(X_j), D_j^{\text{true}})\right] = \mathcal{O}(M)$$

$$\beta_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\beta_j\right)}\sum_{j=1}^{k}\beta_j \Delta q(X_j, \mathcal{T}', \nabla f(X_j), D_j^{\text{true}})\right] \to 0$$

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$$\beta_k = \Theta(1) \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k}(\|\nabla f(X_j) + \nabla c(X_j)^T Y_j^{\text{true}}\|_2 + \|c(X_j)\|_2)\right] = \mathcal{O}(M)$$

$$\beta_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\beta_j\right)}\sum_{j=1}^{k}\beta_j(\|\nabla f(X_j) + \nabla c(X_j)^T Y_j^{\text{true}}\|_2 + \|c(X_j)\|_2)\right] \to 0$$

Numerical results: https://github.com/frankecurtis/StochasticSQP

Stochastic SQP (hard constraints) compared to stochastic subgradient method (soft constraints)

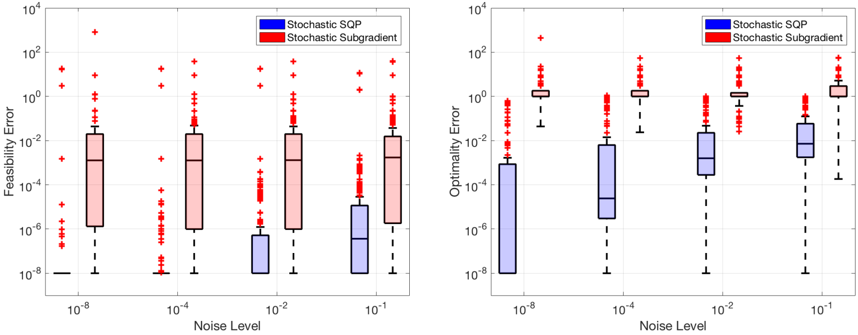


Figure: Box plots for feasibility errors (left) and optimality errors (right).

Outline

Summary

Since our original work, we have considered various extensions.

- stronger convergence guarantees (almost-sure convergence)
- convergence of Lagrange multiplier estimates
- relaxed constraint qualifications
- worst-case complexity guarantees
- generally constrained problems (with inequality constraints as well)
- interior-point methods
- iterative linear system solvers and inexactness

Almost-sure convergence of merit function value

Convergence of the algorithm is driven by the exact merit function

$$\phi_\tau(X) = \tau f(X) + \|c(X)\|$$

Reductions in a local model of ϕ_τ can be tied to a stationarity measure

$$\Delta q_\tau(X, \nabla f(X), H, D^{\text{true}}) \qquad \sim \qquad \|\nabla f(X) + \nabla c(X) Y\|^2 + \|c(X)\|$$

Lemma

Suppose $\mathbb{E}[G_k|\mathcal{F}_k] = \nabla f(X_k)$ and $\mathbb{E}[\|G_k - \nabla f(X_k)|\mathcal{F}_k\|^2] \leq M$. Then, by a classical theorem of Robbins and Siegmund (1971), one finds that, almost surely,

$$\lim_{k\to\infty} \{\phi_\tau(X_k)\} \text{ exists and is finite and}$$

$$\liminf_{k\to\infty} \Delta q_\tau(X_k, \nabla f(X_k), H_k, D_k^{\text{true}}) = 0$$

Almost-sure convergence of the primal iterates

Theorem

Suppose that there exists $x_* \in \mathcal{X}$ *with* $c(x_*) = 0$, $\mu \in \mathbb{R}_{>1}$, *and* $\epsilon \in \mathbb{R}_{>0}$ *such that for all*

$$x \in \mathcal{X}_{\epsilon,x_*} := \{x \in \mathcal{X} : \|x - x_*\|_2 \leq \epsilon\}$$

one finds that

$$\phi_\tau(x) - \phi_\tau(x_*) \begin{cases} = 0 & \text{if } x = x_* \\ \in (0, \mu(\tau\|Z(x)^T \nabla f(x)\|_2^2 + \|c(x)\|_2)] & \text{otherwise,} \end{cases}$$

where for all $x \in \mathcal{X}_{\epsilon,x_*}$ *one defines* $Z(x) \in \mathbb{R}^{n\times(n-m)}$ *as some orthonormal matrix whose columns form a basis for the null space of* $\nabla c(x)^T$. *Then, if* $\limsup_{k\to\infty}\{\|X_k - x_*\|_2\} \leq \epsilon$ *almost surely, it follows that*

$$\{\phi_\tau(X_k)\} \xrightarrow{a.s.} \phi_\tau(x_*), \quad \{X_k\} \xrightarrow{a.s.} x_*, \quad \text{and} \quad \left\{\begin{bmatrix} \nabla f(X_k) + \nabla c(X_k) Y_k^{\text{true}} \\ c(X_k) \end{bmatrix}\right\} \xrightarrow{a.s.} 0.$$

Lagrange multiplier convergence

Theorem

Suppose (x_*, y_*) *is a stationary point. Then, for any* $k \in \mathbb{N}$*, one finds* $\|X_k - x_*\|_2 \leq \epsilon$ *implies*

$$\|Y_k - y_*\|_2 \leq \kappa_y \|X_k - x_*\|_2 + r^{-1} \|\nabla f(X_k) - G_k\|_2$$
$$\textit{and} \;\; \|Y_k^{\text{true}} - y_*\|_2 \leq \kappa_y \|X_k - x_*\|_2 \;\; \textit{for some} \;\; (\kappa, r) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}.$$

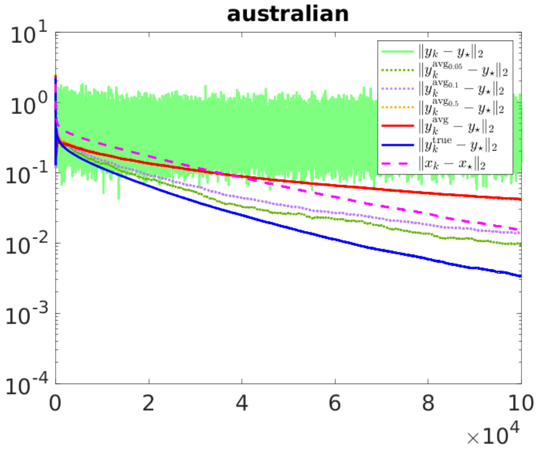
Computed multipliers *always* have error. Consider *averaged* multipliers $\{Y_k^{\text{avg}}\}$ instead.

Theorem

If the iterate sequence converges almost surely to x_**, i.e.,* $\{X_k\} \xrightarrow{a.s.} x_*$*, then*

$$\{Y_k^{\text{true}}\} \xrightarrow{a.s.} y_* \;\; \textit{and} \;\; \{Y_k^{\text{avg}}\} \xrightarrow{a.s.} y_*.$$

Constrained logistic regression: **australian** dataset (LIBSVM)



Complexity of $\mathcal{O}(\epsilon^{-2})$ for deterministic algorithm

All reductions in the merit function can be cast in terms of smallest τ.

Since $\tau_{\min}$ is determined by the initial point, *it will be reached.*

Theorem

For any $\epsilon \in (0,1)$, there exists $(\kappa_1, \kappa_2) \in (0,\infty) \times (0,\infty)$ such that

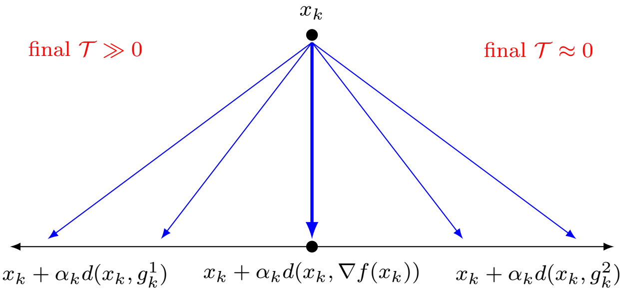
$$\|\nabla f(x_k) + J_k^T y_k\| \leq \epsilon \text{ and } \sqrt{\|c_k\|_1} \leq \epsilon$$

in a number of iterations no more than

$$\left(\frac{\tau_0(f_1 - f_{\inf}) + \|c_1\|_1}{\min\{\kappa_1, \kappa_2 \tau_{\min}\}}\right)\epsilon^{-2}.$$

Challenge in the stochastic setting

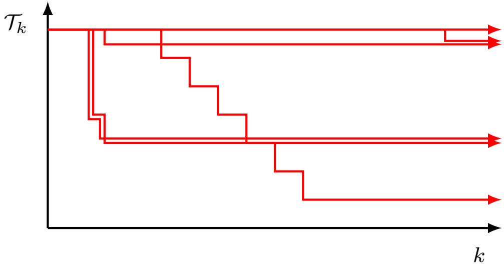
We are minimizing a function that is changing during the optimization.



Challenge in the stochastic setting

In the stochastic setting, minimum $\mathcal{T}$ is not determined by the initial point.

- Even if we assume $\mathcal{T}_k \geq \tau_{\min} > 0$ for all k in all realizations, the final $\mathcal{T}$ is not determined.
- This means we cannot cast all reductions in terms of some fixed constant τ.



Our approach

In fact, $\mathcal{T}$ reaching some minimum value is not necessary.

- Important: Diminishing probability of continued imbalance between "true" merit parameter update and "stochastic" merit parameter update.
- In iteration k, the algorithm has obtained the merit parameter value $\mathcal{T}_{k-1}$.
- If the true gradient is computed, then one obtains $\mathcal{T}_k^{\text{trial,true}}$.

Lemma

Suppose that the merit parameter is reduced at most $s_{\max}$ times. For any $\delta \in (0,1)$, one finds that

$$\mathbb{P}\left[|\{k : \mathcal{T}_k^{trial,true} < \mathcal{T}_{k-1}\}| \leq \left\lceil \frac{\ell(s_{\max},\delta)}{p} \right\rceil\right] \geq 1-\delta,$$

where $p \in (0,1)$ (related to a bounded imbalance assumption we make) and

$$\ell(s_{\max},\delta) := s_{\max} + \log(1/\delta) + \sqrt{\log(1/\delta)^2 + 2s_{\max}\log(1/\delta)} > 0.$$

▸ Details

Worst-case iteration complexity of $\widetilde{\mathcal{O}}(\epsilon^{-4})$

Theorem

Suppose the algorithm is run $k_{\max}$ iterations with $\beta_k = \gamma/\sqrt{k_{\max}+1}$ and

- *the merit parameter is reduced at most $s_{\max} \in \{0,1,\dots,k_{\max}\}$ times.*

Let k_* be sampled uniformly over $\{1,\dots,k_{\max}\}$. Then, with probability $1-\delta$,

$$\mathbb{E}[\|\nabla f(X_{k_*}) + J(X_{k_*})^T Y_{k_*}\|_2^2 + \|c(X_{k_*})\|_1] \leq \frac{\tau_{-1}(f_0 - f_{\inf}) + \|c_0\|_1 + M}{\sqrt{k_{\max}+1}} + \frac{(\tau_{-1} - \tau_{\min})(s_{\max}\log(k_{\max}) + \log(1/\delta))}{\sqrt{k_{\max}+1}}$$

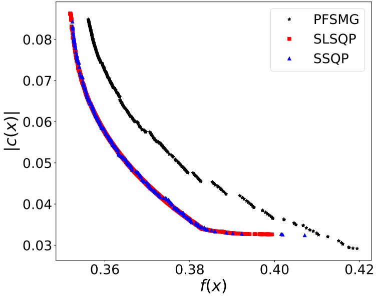
Theorem

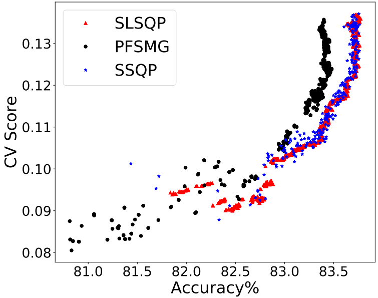
If the stochastic gradient estimates are sub-Gaussian, then with probabiliy $1-\bar{\delta}$

$$s_{\max} = \mathcal{O}\left(\log\left(\log\left(\frac{k_{\max}}{\bar{\delta}}\right)\right)\right).$$

Inequality-constrained: Fair learning

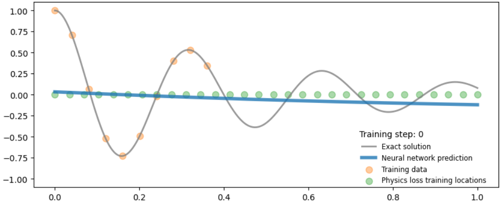
Employed in an ϵ-constraint method for fair machine learning:

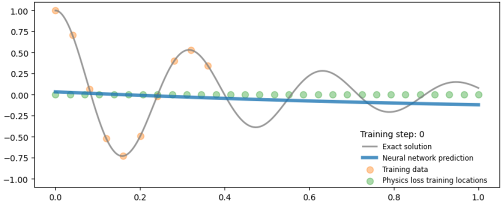




$$\min_{x\in\mathbb{R}^n} \frac{1}{N^o} \sum_{(v_i,y_i)\in D_o} \ell(x, v_i, y_i) \quad \text{s.t.} \quad -\epsilon \leq \frac{1}{N^c} \sum_{(v_i,a_i)\in D_c} (a_i - \overline{a})x^T v_i \leq \epsilon$$

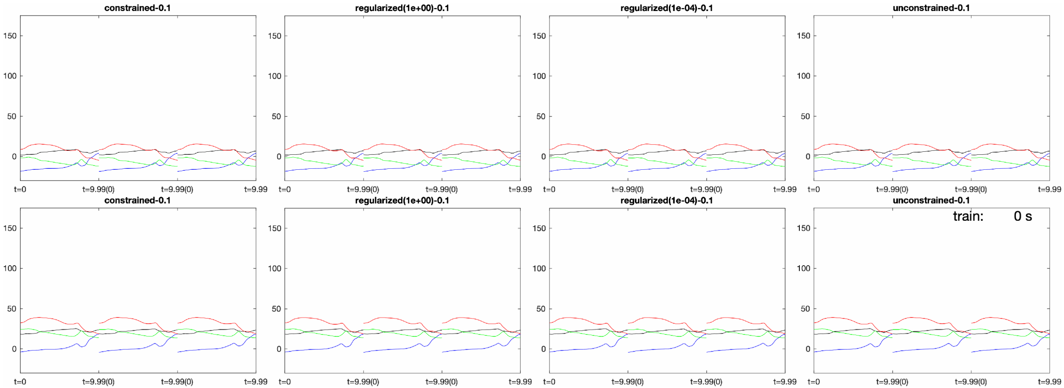
Physics-informed learning





Problem from https://benmoseley.blog/blog/

Mass-balance



Thank you!

Questions?

Outline

Details

Back

Some details on the tree construction for our complexity analysis...

Chernoff bound

How do we get there?

Lemma (Chernoff bound, multiplicative form)

Let $\{Y_0, \ldots, Y_k\}$ be independent Bernoulli random variables. Then, for any $s_{\max} \in \mathbb{N}$ and $\delta \in (0,1)$,

$$\sum_{j=0}^{k} \mathbb{P}[Y_j = 1] \geq \ell(s_{\max}, \delta) \implies \mathbb{P}\left[\sum_{j=0}^{k} Y_j \leq s_{\max}\right] \leq \delta.$$

We construct a tree whose nodes are signatures of possible runs of the algorithm.

- A realization $\{g_0, \ldots, g_k\}$ belongs to a node if and only if a certain number of decreases of $\mathcal{T}$ have occurred and the probability of decrease in the current iteration is in a given closed/open interval.
- Bad leaves are those when the probability of decrease has accumulated beyond a threshold, yet the merit parameter has not been decreased sufficiently often.
- Along the way, we apply a Chernoff bound on a carefully constructed set of (independent Bernoulli) random variables to bound probabilities associated with bad leaves.

Node definition

Let $[k] := \{0, 1, \ldots, k\}$ and define

- $p_{[k]}$ = probabilities of merit parameter decreases
- $w_{[k]}$ = counter of merit parameter decreases

Then, define nodes of the tree according to

$$\boxed{G_{[k-1]} \in N(p_{[k]}, w_{[k]})}$$

if and only if

$$\boxed{\begin{aligned} G_{[k-2]} &\in N(p_{[k-1]}, w_{[k-1]}) \\ \mathbb{P}[\mathcal{T}_k < \mathcal{T}_{k-1} | \mathcal{F}_k] &\in \iota(p_k) \\ \sum_{i=1}^{k-1} \mathbb{1}[\mathcal{T}_i < \mathcal{T}_{i-1}] &= w_k \end{aligned}}$$

Visualization