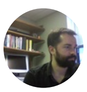
Central Groupoids, Central Digraphs, and Zero-One Matrices A Satisfying A^2=J

Article *in* Journal of Combinatorial Theory Series A · January 2004

DOI: 10.1016/j.jcta.2003.10.001 · Source: DBLP

4 authors, including:



Frank E. Curtis
Lehigh University

Chi-Kwong Li
College of William and Mary

Daniel Pragel
Georgia Gwinnett College

Central Groupoids, Central Digraphs, and Zero-One Matrices A Satisfying $A^2 = J$

Frank Curtis [*], John Drew[†], Chi-Kwong Li[‡], and Daniel Pragel[§]

September 25, 2003

Abstract

We study central groupoids, central digraphs, and zero-one matrices A satisfying $A^2 = J$. A survey of known results is given, including short proofs for some of them; new results and techniques are developed, and conjectures are settled. Open questions and new conjectures are mentioned.

[*]Department of Mathematics, College of William and Mary, PO Box 8795, Williamsburg, VA 23187-8795. Current address: School of Engineering and Applied Science, Northwestern University, Evanston, IL 60208-3100 (f-curtis@northwestern.edu)

[†]Department of Mathematics, College of William and Mary, PO Box 8795, Williamsburg, VA 23187-8795. (jhdrew@math.wm.edu)

[‡]Department of Mathematics, College of William and Mary, PO Box 8795, Williamsburg, VA 23187-8795. (ckli@math.wm.edu)

[§]Department of Mathematics, SUNY at Albany, Albany, NY 12222. Current address: Department of Mathematics, California Institute of Technology, Pasadena, CA 91125 (rundmp@rochester.rr.com)

1 Introduction

A directed graph with n vertices is called a *central directed graph* (*central digraph*) if there is a unique length two walk between every pair of vertices. In terms of the adjacency matrix A of the digraph, this is equivalent to $A^2 = J_n$, where J_n is the $n \times n$ matrix of all ones. In our discussion, we will let $\mathcal{A}_n$ be the set of all such matrices. These concepts are related to the algebraic structure known as a *central groupoid*, which is a non-empty set S and a binary operation $*$ such that

$$(x * y) * (y * z) = y$$

for all elements x, y, z in the set. One can establish the correspondence between a central digraph and a central groupoid as follows. Identify the vertices of the digraph with the elements in the groupoid so that $i * j = k$ if and only if $i \to k \to j$ is the unique length two path from i to j.

Central digraphs, central groupoids and the matrix equation $A^2 = J_n$ have attracted the attention of many researchers in different areas because of their very rich and beautiful algebraic and combinatorial structures, and their connection to other pure and applied problems; e.g., see [5, 6, 7, 8] and their references.

In [5], Hoffman raised the question of finding all matrices in $\mathcal{A}_n$, which seems to be a very difficult problem. Nevertheless, many interesting properties and techniques have been discovered. The purpose of this paper is to obtain additional results on these subjects and to further develop techniques and insights in studying these concepts. In particular, we shall use approaches from algebra, combinatorial theory, matrix theory, and scientific computations. Our paper is organized as follows. In section 2, we reprove most of the basic results and obtain several new results. In section 3, we consider a simple operation for transforming a matrix in $\mathcal{A}_n$ to another matrix in $\mathcal{A}_n$. Using this operation repeatedly, we give a short proof for the interesting result of Shader [7], which lists the ranks achieved by matrices in $\mathcal{A}_n$. In section 4, we study sub-central groupoids, i.e., subsets of a central groupoid which are themselves central groupoids. We will mention many open problems and conjectures.

Denote by $\{e_1, \ldots, e_k\}$ the standard basis for $\mathbf{R}^k$, with $\mathbf{1}_k = e_1 + \cdots + e_k$. Suppose $n = k^2$. Then the *standard matrix* A_0 in $\mathcal{A}_n$ is the $k \times k$ block matrix whose (i, j) block is $A_{ij} = e_j \mathbf{1}_k^t$. For example, if $k = 3$, then $A_0 = (A_{ij})_{1 \le i,j \le 3} \in \mathcal{A}_9$ with

$$A_{i1} = e_1 \mathbf{1}_3^t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{i2} = e_2 \mathbf{1}_3^t = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A_{i3} = e_3 \mathbf{1}_3^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

for $i = 1, 2, 3$. In the following we always use A_0 to denote the standard matrix in $\mathcal{A}_{k^2}$.

Our paper is based on the REU report [2], which contains some computer programs developed for the study of this topic. The report and computer programs are available at http://www.resnet.wm.edu/~cklixx/mathlib.html

2 Basic Properties

In this section, we collect known results on $\mathcal{A}_n$ (for example, see [6, 7]) and give short proofs for some of them. After that, we will establish several new results.

Theorem 2.1 *Let n be a positive integer. Then $\mathcal{A}_n \neq \emptyset$ if and only if $n = k^2$ for some positive integer k. Furthermore, if $n = k^2$ and $A \in \mathcal{A}_n$, then*

(a) *all row sums and column sums of A equal k,*

(b) *A has eigenvalues $k, 0, \ldots, 0$.*

(c) *A has exactly k 1's on its main diagonal.*

Proof. Suppose $n = k^2$. Then it is easy to verify that the $n \times n$ standard matrix A_0 defined in Section 1 belongs to $\mathcal{A}_n$.

Conversely, suppose $\mathcal{A}_n$ is non-empty, and $A \in \mathcal{A}_n$. If A has row sums $r_1, \ldots, r_n$ and column sums $c_1, \ldots, c_n$, then $A^3 = AA^2 = AJ = (r_1, \ldots, r_n)^t \mathbf{1}_n^t$ and $A^3 = A^2 A = JA = \mathbf{1}_n(c_1, \cdots, c_n)$. Thus, all row sums and column sums are the same, say, equal to k. In particular, A has Perron root k with a positive (Perron) eigenvector $\mathbf{1}_{k^2}$.

Now, since A^2 has eigenvalues $n, 0, \ldots, 0$, A has eigenvalues $\sqrt{n}, 0, \ldots, 0$, and hence $k = \sqrt{n}$. Since the trace of a matrix is equal to the sum of its eigenvalues, A has exactly k 1's on its main diagonal. $\square$

It is easy to check that up to permutation similarity, there is only one element, namely, the standard matrix, in $\mathcal{A}_4$. In [6], the author announced that up to permutation similarity, there are 6 distinct elements in $\mathcal{A}_9$; see [2] for a proof. They are:

$$A_0 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&1&0&0&0&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&1&0&0&0&1&1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&1&0&0&0&1&1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&1&0&0&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&0\\ 0&0&1&0&0&0&1&1&0\\ 1&0&1&0&1&0&0&0&0\\ 0&1&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&1&0&0&0&1&0&0&0\\ 0&0&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&0&1&0&0&0&0&1&0\\ 0&0&0&1&1&1&0&0&0\\ 0&1&0&0&0&0&1&0&1 \end{bmatrix}.$$

The problem of determining all the matrices in $\mathcal{A}_{k^2}$ for $k \ge 4$ raised in [5] remains open.

Theorem 2.2 *The Jordan forms of matrices in $\mathcal{A}_n$ are precisely:*

$$[\sqrt{n}] \oplus \underbrace{B \oplus \cdots \oplus B}_{p} \oplus 0_{n-2p-1} \qquad \textit{with} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\sqrt{n} - 1 \le p \le (n-1)/2$. Consequently, if $A \in \mathcal{A}_n$, then

$$\sqrt{n} \le \operatorname{rank}(A) \le (n+1)/2.$$

The first equality holds if and only if A is permutationally similar to the standard matrix A_0 defined in Section 1.

Proof. Since A^2 is diagonalizable with eigenvalues $n, 0, \ldots, 0$, the Jordan blocks of A corresponding to the eigenvalue 0 have size at most 2. Thus, the rank of A is at most $(n+1)/2$. By the result in [7] (see also Theorem 3.3 in the next section), all such ranks can be attained, and thus the prescribed Jordan structure can be attained.

Now, to prove the last assertion, note that AA^t is a symmetric positive semi-definite, nonnegative matrix with Perron vector $\mathbf{1}_{k^2}$ and trace k^3, and hence each eigenvalue of AA^t is nonnegative, and at most k^2. It follows that AA^t has at least k positive eigenvalues. Moreover, if there are exactly k positive eigenvalues, then each of them is equal to the Perron root k^2, and AA^t has rank k. So, there is a permutation matrix P such that $PAA^tP^t = A_1 \oplus \cdots \oplus A_k$, where each A_j has rank 1, all diagonal entries equal to k, and one nonzero eigenvalues equal to k^2. It follows that $A_j = kJ_k$ for $j = 1, \ldots, k$. Evidently, this happens if and only if PA has k groups of k identical rows, or equivalently, A is permutationally similar to the standard matrix. □

A matrix $A \in \mathcal{A}_n$ is said to have row multiplicities $m_1 \ge m_2 \ge \cdots \ge m_s$ if A has m_1 rows that are equal, m_2 other rows that are equal, etc., where $m_1 + m_2 + \cdots + m_s = n$. Similarly we can define the column multiplicities of A.

R.R. Fletcher III conjectured (see [3, 4]) that for each A such that $A^2 = J$, the multi-sets of integers representing the column and row multiplicities of A are equal. We have settled his conjecture by the following result:

Theorem 2.3 *Suppose $n = k^2$ for some positive integer k. If $k \leq 3$ and $A \in \mathcal{A}_n$, then the row and column multiplicities of A are the same. If $k \geq 4$, there there is an $A \in \mathcal{A}_n$ whose row and column multiplicities are different.*

Proof. If $n = 1, 4$, the result is clear. If $n = 9$, the result is true by the comment in [6]. Suppose $n = k^2 \geq 16$. Consider $A = (A_{ij})_{1 \leq i,j \leq k}$ such that $A_{i1} = e_1(e_1 + e_k)^t + e_2(e_2 + \cdots + e_{k-1})^t$ and $A_{i2} = e_2(e_1 + e_k)^t + e_1(e_2 + \cdots + e_{k-1})^t$ for $i = 1, \ldots, k-1$, and $A_{ij} = e_j \mathbf{1}_k^t$ for all other $i, j \in \{1, \ldots, k\}$. Then A has row multiplicities $\underbrace{k, \ldots, k}_{k-2}, k-1, k-1, 1, 1$, and column multiplicities $\underbrace{k, \ldots, k}_{k-2}, k-2, k-2, 2, 2$. The result follows. $\square$

For example, when $k = 4$, the construction in the above proof yields

$$\begin{bmatrix} B_1 & B_2 & A_3 & A_4 \\ B_1 & B_2 & A_3 & A_4 \\ B_1 & B_2 & A_3 & A_4 \\ A_1 & A_2 & A_3 & A_4 \end{bmatrix}, \quad \text{where} \quad B_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $A_j = e_j \mathbf{1}_k^t$ for $j = 1, \ldots, 4$. This matrix has row multiplicities {4,4,3,3,1,1} and column multiplicities {4,4,2,2,2,2}.

Next, we turn to some additional new results on $\mathcal{A}_n$. First, we present an additive decomposition of matrices in $\mathcal{A}_n$ in terms of permutation matrices. The following lemma will give some additional structure to the summands of the decomposition.

Lemma 2.4 *Let $A \in \mathcal{A}_n$ with $n = k^2$ for some positive integer k, and let $G(A)$ be the directed graph of A with vertices $v_1, \ldots, v_n$. Then k of the vertices in $G(A)$ have loops, while all other vertices in $G(A)$ are paired in two-cycles, with no vertex belonging to more than one two-cycle.*

Proof. Since A has k 1's on its main diagonal, $G(A)$ has k loops. No other vertex can be in a two-cycle with a loop vertex (that is, one with a loop) since then there would be two length two walks from the idempotent vertex to itself. Since there must be a length two walk from each non-idempotent vertex to itself, the walk must be a two-cycle with two non-idempotent vertices. No vertex v_i can be in two two-cycles, say with v_j and v_k, since then there would be at least two length two walks from v_j to v_k. $\square$

Theorem 2.5 *Let $n = k^2$ for some positive integer k. Every $A \in \mathcal{A}_n$ can be written as the sum of k permutation matrices, $A = P_1 + \cdots + P_k$, and in any such decomposition $P_i P_j$ and $P_r P_s$ have no common nonzero entries for any $(i, j) \neq (r, s)$ with $1 \leq i, j, r, s \leq k$. Moreover, we may assume that P_1 satisfies $P_1^2 = I_n$.*

Proof. By [1, Corollary 1.2.5], A is the sum of k permutation matrices. Since $A^2 = J_n$, the condition on P_iP_j follows.

Finally, we assume that P_1 is the adjacency matrix corresponding to the loops and two cycles in Lemma 2.4. Then $P_1^2 = I_n$ and $A - P_1$ is a $(0, 1)$ matrix with all row sums and column sums $k - 1$. By [1, Corollary 1.2.5] again, we can decompose $A - P_1$ as the sum of $k - 1$ permutation matrices. $\square$

A word $W(A, A^t)$ of length m is defined to be a product of m matrices $X_1, \ldots, X_m$ such that $X_j \in \{A, A^t\}$ for all j. The following proposition is new.

Theorem 2.6 *Suppose $A \in \mathcal{A}_n$, and $W(A, A^t)$ is a word of length m not of the form $(AA^t)^{m/2}$ or $(A^tA)^{m/2}$ where m is even. Then $W(A, A^t)$ has eigenvalues $k^m, 0, \ldots, 0$.*

Proof. If $A^2 = J$ or $(A^t)^2 = J$ appears in $W(A, A^t)$, then $W(A, A^t) = k^{m-2}J$, and we are done. If not, $W(A, A^t)$ must be of the form $(AA^t)^{(m-1)/2}A$ or $(A^tA)^{(m-1)/2}A^t$ by our assumption. By the fact that XY and YX have the same eigenvalues for any square matrices X and Y of the same size, in both cases we can shift the first letter to the last letter in the word to obtain a new word with the same eigenvalues. The result now follows from the first case. $\square$

Note that the number of nonzero eigenvalues of AA^t represents the rank of A. Also, partitioning the set $\mathcal{A}_n$ according to ranks or different singular values may be useful in constructing or counting part or all the matrices in $\mathcal{A}_n$. So, we pose the following.

Problem 2.7 Determine all possible eigenvalues of AA^t for $A \in \mathcal{A}_n$.

3 Transforming Matrices in $\mathcal{A}_n$ by Switches

Suppose $(i_1, \cdots, i_k)$ and $(j_1, \ldots, j_k)$ are subsequences of $(1, \ldots, n)$. Denote by

$$A[i_1, \ldots, i_k; j_1, \ldots, j_k]$$

the submatrix of A lying in rows $i_1, \ldots, i_k$, and columns $j_1, \ldots, j_k$. We have the following observation, where the first part was proved in [3].

Theorem 3.1 *Let $A \in \mathcal{A}_n$, $1 \le p < q \le n$, and $1 \le r, s \le n$ satisfy*

$$A[p, q; r, s] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose $\tilde{A}$ is obtained from A by replacing $A[p, q; r, s]$ with $J_2 - A[p, q; r, s]$. Then $\tilde{A} \in \mathcal{A}_n$ if and only if $\{p, q\} \cap \{r, s\} = \emptyset$, rows r and s of A are the same, and columns p and q of A are the same. Moreover, the ranks of $\tilde{A}$ and A differ by at most one.

Proof. The first assertion was proved in [3] using the following graph theory argument. Let D and $\tilde{D}$ be the digraphs corresponding to the matrices A and $\tilde{A}$. Then $\tilde{D}$ is obtained from D by replacing the arcs (p, r) and (q, s) by (p, s) and (q, r).

Suppose $A, \tilde{A} \in \mathcal{A}_n$. If $\{p, q\} \cap \{r, s\}$ has j elements with $j \in \{1, 2\}$, then changing D to $\tilde{D}$ will create or destroy j loops. Thus, $\tilde{D}$ does not have $\sqrt{n}$ loops and cannot be a central digraph.

Now, for any vertex i, consider the length 2 walks from i to r in D and $\tilde{D}$. We see that (i, p) is an edge in D if and only if (i, q) is an edge in $\tilde{D}$. Thus, the pth column of A is the same as the qth column of $\tilde{A}$, which is the same as that of A as $q \notin \{r, s\}$. Similarly, for any vertex j, considering the length 2 walks, from p to j in D and $\tilde{D}$, we see that the rth and sth rows of A are the same.

One can readily verify the converse by graph theory or matrix theory consideration.

For the second assertion, note that $\tilde{A} - A = \pm(e_p - e_q)(e_r - e_s)^t$, and thus the ranks of $\tilde{A}$ and A can differ at most by one. □

If $\tilde{A} \in \mathcal{A}_n$ is obtained from A by a change described in the above proposition, we say that $\tilde{A}$ is obtained from A by a *switch*. The following conjecture is mentioned in [3] (see also [4]).

Conjecture 3.2 Every $A \in \mathcal{A}_n$ can be obtained from the standard matrix by a finite number of switches.

While we are not able to prove or disprove the above conjecture, we can use the concept of switches to give a short proof of the interesting result in [7], namely, there are matrices in $\mathcal{A}_n$ with rank r if and only if $\sqrt{n} \le r \le (n+1)/2$. In [7], the proof was done by several delicate constructions according to the ranks of the matrices in $\mathcal{A}_n$. One needs to check a rather long list of conditions to conclude that the constructed matrices are indeed in $\mathcal{A}_n$. In Theorem 3.3, we will describe how one can use switches to transform the standard matrix in $\mathcal{A}_{k^2}$ to a matrix with maximum rank r in $r - k + 1$ steps. Since a switch can only increase the rank of a matrix in $\mathcal{A}_n$ by at most one by Theorem 3.1, it follows that the rank of the matrix in our construction is increased by one in each step. In the following, we will denote a switch by $[p, q; r, s]$, and the basic circulant by $T_k = [e_k|e_1|e_2|\cdots|e_{k-1}]$; e.g.,

$$T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; T_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.3 *Consider the following three types of switches on the standard matrix* $A_0 = (A_{ij})_{1 \le i,j \le k} \in \mathcal{A}_n$, *in which* $n = k^2$ *with* $k \ge 3$*:*

(1) $(k-2)$ *switches in the submatrix* $[A_{11}|A_{12}|\cdots|A_{1k}]$ *as follows:*

$[2,3;k+3,2k+3]$,

$[3,4;2k+4,3k+4]$,

$[4,5;3k+5,4k+5]$,

$\vdots \quad \vdots \quad \vdots$

$[k-2,k-1;(k-3)k+(k-1),(k-2)k+(k-1)]$,

$[k-1,k;(k-2)k+k,(k-1)k+k]$.

(2) *For* $k \geq 4$, $(k-2)(k-3)/2$ *switches in the submatrices* $[A_{i1}|A_{i2}|\cdots|A_{ik}]$ *for* $i=4,\ldots,k$ *as follows. For* $i=4,\ldots,k$, *we have* $(i-3)$ *switches:*

$[(i-1)k+2,(i-1)(k+1);k+i,(i-2)k+i]$,

$[(i-1)k+2,(i-1)k+3;2k+i,(i-2)k+i]$,

$[(i-1)k+3,(i-1)k+4;3k+i,(i-2)k+i]$,

$\vdots \quad \vdots \quad \vdots$

$[(i-1)k+(i-3),(i-1)k+(i-2);(i-3)k+i,(i-2)k+i]$

(*here ignoring the first switch, one has an easy pattern for the other* $i-4$ *switches) that transform the submatrix* I_{i-2} *in rows indexed by* $2,\ldots,i-1$ *and columns indexed by* $k+i,2k+i,\ldots,(i-2)k+i$ *of* $[A_{i1}|A_{i2}|\cdots|A_{ik}]$ *to* T_{i-2}.

For example, there is 1 switch for $i=4$*:* $[3k+2,3k+3;k+4,2k+4]$*; there are 2 switches for* $i=5$*:* $[4k+2,4k+4;1k+5,3k+5]$, $[4k+2,4k+3;2k+5,3k+5]$*; there are 3 switches for* $i=6$*:* $[5k+2,5k+5;1k+6,4k+6]$, $[5k+2,5k+3;2k+6,4k+6]$, $[5k+3,5k+4;3k+6,4k+6]$.

(3) $[(k-1)/2]$ *switches in the submatrices* $[A_{i1}|A_{i2}|\cdots|A_{ik}]$ *for* $i=3,5,7,\ldots$ *as follows:*

$[2k+1,2(k+1);1,k+1]$,

$[4k+1,4(k+1);1,3k+1]$,

$\vdots \quad \vdots \quad \vdots$

$[2mk+1,2m(k+1);1,(2m-1)k+1]$, *where* $m=[(k-1)/2]$.

If A_0 *is modified by consecutively applying the above switches, then the following conditions hold:*

(a) *Every switch is legal, i.e., the matrix remains in* $\mathcal{A}_n$ *after every switch.*

(b) *Every switch increases the rank of the current matrix by one, and the final matrix has rank* $[n/2]$.

Proof. To prove part (a), note that by Theorem 3.1, a switch $[p, q; r, s]$ may be legally performed on a matrix $B \in \mathcal{A}_n$ if and only if $\{p, q\} \cap \{r, s\} = \emptyset$, row r and row s of B are identical, and column p and column q of B are identical.

In the proposed switches, rows with the following indices will be changed:

Type (1) switches: $2, \ldots, k$,

Type (2) switches:

$$\begin{array}{l} 3k+2, 3k+3 \\ 4k+2, 4k+3, 4k+4, \\ 5k+2, 5k+3, 5k+4, 5k+5, \\ \quad\vdots \quad \vdots \quad \vdots \\ (k-1)k+2, (k-1)k+3, \ldots, (k-1)k+(k-1). \end{array}$$

Type (3) switches:

$$\begin{array}{l} 2k+1, 2(k+1) \\ 4k+1, 4(k+1) \\ 6k+1, 6(k+1) \\ \quad\vdots \quad \vdots \quad \vdots \\ 2mk+1, 2m(k+1). \end{array}$$

Also, columns with the following indices will be changed:

Type (1) switches:

$$\begin{array}{l} k+3 \\ 2k+3, 2k+4, \\ 3k+4, 3k+5, \\ \quad\vdots \quad \vdots \quad \vdots \\ (k-2)k+(k-1), (k-2)k+k, \\ (k-1)k+k. \end{array}$$

Type (2) switches:

$$\begin{array}{l} k+4, 2k+4 \\ k+5, 2k+5, 3k+5, \\ \quad\vdots \quad \vdots \quad \vdots \\ k+(k-1), 2k+(k-1), \ldots, (k-3)k+(k-1). \\ k+k, 2k+k, \ldots, (k-2)k+k. \end{array}$$

Type (3) switches: $1, k+1, 3k+1, \ldots, (2m-1)k+1$.

Note that the two lists of indices corresponding to row changes and columns changes are disjoint. Hence, for each proposed switch $[p, q; r, s]$, no matter how many other proposed switches have already been performed, columns p and q have not been changed and they are identical to columns p and q in the original matrix A_0, which are equal; similarly, rows r and s have not been changed throughout the process and thus are identical to rows r and s in the original matrix A_0, which are equal. Therefore, all the proposed switches are legal.

Despite this simple proof of (a), one may want to trace the pattern of how the list of the forbidden (for use in switches) row and columns indices grows throughout the process, but still leaves an adequate number of equal rows and equal columns for future switches.

To prove (b), note that the total number of proposed switches equals

$$(k-2)+(k-2)(k-3)/2+[(k-1)/2]=(k-1)(k-2)/2+[(k-1)/2].$$

Since every switch can only increase rank by at most one, the final matrix has rank not larger than

$$k+(k-1)(k-2)/2+[(k-1)/2]=\begin{cases}(k^2+1)/2 & \text{if } k \text{ is odd,}\\ k^2/2 & \text{if } k \text{ is even,}\end{cases}$$

which is $[(n+1)/2]$. Thus, we need only to show that the final matrix has rank $[(n+1)/2]$; it will then follow that each proposed switch indeed increases the rank by one. To achieve our goal, we show that the row space of the final matrix contains $[(n+1)/2]$ linearly independent vectors.

First, pick the k rows in $[A_{21}|A_{22}|\cdots|A_{2k}]$, which have not been changed at all. Denote these row vectors by $u_1,\ldots,u_k$. Note that the column indices of the leading ones of these row vectors are:

$$1, k+1, 2k+1, \ldots, (k-1)k+1.$$

Next, pick the $(k-2)$ rows that resulted from the type (1) switches, namely, those rows indexed by $3,4,\ldots,k$. Denote these row vectors by $v_1,\ldots,v_{k-2}$. Note that the column indices of the leading ones of these rows are:

$$k+3, 2k+4, 3k+5, \ldots, (k-2)k+k.$$

Now, consider the rows that resulted from the type (2) switches in $[A_{i1}|\cdots|A_{ik}]$ for $i=4,\ldots,k$.

For $i=4$, consider the row indexed by $3k+2$ in the final matrix. Subtracting this vector from u_2 (the second row of $[A_{21}|\cdots|A_{2k}]$), we get a row vector with leading one at the $(k+4)$th position. Denote this vector by w_1.

For $i=5$, consider the rows indexed by $4k+2$ and $4k+3$ in the final matrix. Subtracting these vectors from u_2 and u_3 (the second and third row of $[A_{21}|\cdots|A_{2k}]$), respectively, we get two row vectors with leading ones at the positions indexed by $k+5, 2k+5$. Denote these vectors by w_2 and w_3.

For $i=6$, consider the rows indexed by $5k+2$, $5k+3$, and $5k+4$ in the final matrix. Subtracting these vectors from u_2, u_3 and u_4, respectively, we get three row vectors with leading ones in the positions indexed by $k+6, 2k+6, 3k+6$. Denote these vectors by w_4, w_5 and w_6.

Continuing the above method, we obtain vectors $w_1,\ldots,w_r$ in the row space of the final matrix, where $r=(k-2)(k-3)/2$, and the leading ones of these vectors are in the positions:

$$
\begin{array}{l}
k+4, \\
k+5, 2k+5, \\
k+6, 2k+6, 3k+6, \\
\qquad \vdots \qquad \vdots \\
2k, 3k, \ldots, (k-2)k.
\end{array}
$$

Finally, consider the following row vectors in the final matrix that resulted from type (3) switches, namely, those rows indexed by

$$2k+1, 4k+1, 6k+1, \ldots, 2mk+1,$$

where $m = [(k-1)/2]$. Let $z_1, \ldots, z_m$ be these row vectors. Let $x_1 = z_m$ and $x_{j+1} = z_{j+1} - z_j + u_{2j}$ for $j = 1, \ldots, m-1$. Then $x_1, \ldots, x_m$ are vectors in the row space of the final matrix, and their leading ones lie in the positions indexed by

$$2, k+2, 3k+2, 5k+2, \ldots, (2m-3)k+2.$$

Now, consider the vectors

$$u_1, \ldots, u_k, v_1, \ldots, v_{k-2}, w_1, \ldots, w_r, x_1, \ldots, x_m$$

in the row space of the final matrix. Since their leading ones all lie in different positions, we have a linearly independent set of size $[(n+1)/2]$ as desired. $\square$

In the following, we exhibit the matrices with maximum rank in $\mathcal{A}_{16}$ and $\mathcal{A}_{25}$ obtained using our scheme. For easy reference, we highlight the entries involved in the switches we applied. One can follow our proof to identify the basis for the row space in each case.

$$
\begin{bmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&\mathbf{0}&1&0&0&\mathbf{1}&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&1&1&\mathbf{0}&\mathbf{0}&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&1&1&1&\mathbf{0}\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
\mathbf{0}&1&1&1&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0\\
\mathbf{1}&0&0&0&\mathbf{0}&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&\mathbf{0}&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{1}&1&1&1&\mathbf{0}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&\mathbf{0}&1&1&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{1}&0&0&1&1&\mathbf{0}&\mathbf{0}&1&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&1&1&1&\mathbf{0}&\mathbf{0}&0&0&0&0&\mathbf{1}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&1&1&1&1&\mathbf{0}\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
\mathbf{0}&1&1&1&1&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\mathbf{1}&0&0&0&0&\mathbf{0}&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&\mathbf{0}&1&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{1}&0&1&1&1&\mathbf{0}&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
\mathbf{0}&1&1&1&1&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&\mathbf{0}&0&0&0&0&\mathbf{1}&0&0&0&0&\mathbf{0}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&\mathbf{0}&0&0&0&0&\mathbf{1}&0&0&0&0&0\\
\mathbf{1}&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&\mathbf{0}&1&1&1&\mathbf{0}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1
\end{bmatrix}
$$

Next, we consider different matrices in $\mathcal{A}_n$ that can be obtained by one switch.

Proposition 3.4 *Suppose $n = k^2 \geq 9$. Then up to permutation similarity, there are three different matrices obtained from the standard matrix by applying one switch.*

Proof. Let $A \in \mathcal{A}_n$ be the standard matrix. Assume the switch takes place at $A[p, q; r, s]$. Then by Theorem 3.1, $\{p, q\} \cap \{r, s\} = \emptyset$, row r and row s of A are identical, column p and column q of A are identical. We consider two cases.

Case 1. Suppose row r or row s contains a nonzero diagonal entry. In this case, we may apply a permutation similarity and assume that $r = 1$ and $s = k + 1$. Since column p and column q are identical, there exists $m \in \{1, \ldots, k\}$ such that $(m - 1)k < p, q \leq mk$. Now, it is easy to check that $m \neq 1, 2$. If $m \geq 3$, then we may assume that $m = 3$; otherwise, apply a permutation similarity to A involving rows and columns with indices larger than $2k$. Now, in order to have

$$A[p, q; r, s] \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

we see that $(p, q) = (2k + 1, 2k + 2)$. So, up to permutation similarity, there is only one matrix with the desired property in this case.

Case 2. Neither row r nor row s contains a nonzero diagonal entry. By permutation similarity, we may assume that $r = 3$ and $s = k + 3$. Since column p and column q are identical, there exists $m \in \{1, \ldots, k\}$ such that $(m - 1)k < p, q \leq mk$.

If $m = 1$, then $(p, q) = (1, 2)$. Applying the switch $[1, 2; 3, k + 3]$, we get a matrix in $\mathcal{A}_{k^2}$ whose first row has multiplicity one, and has a diagonal entry. Thus, it is not permutationally similar to the matrix obtained in Case 1 using the switch $[2k + 1, 2k + 2; 1, k + 1]$ because it does not have a row with multiplicity one containing a diagonal entry.

If $m = 2$, then $(p, q) = (k+1, k+2)$. Applying this switch, namely, $[k+1, k+2; 3, k+3]$, we get a matrix that is permutationally similar to the one obtained by the switch $[1, 2; 3, k + 3]$ in the preceding paragraph. For example, one can interchange the rows and columns indexed by 1 and 2 with those indexed by $k + 1$ and $k + 2$ to convert one matrix to the other.

Finally, if $m \geq 3$, we may assume that $(m, p, q) = (3, 2k + 1, 2k + 2)$ up to permutation similarity. Applying this switch, we get a matrix that has two columns with multiplicity one and two rows with multiplicity one such that none of these two rows or two columns contains a diagonal entry. So, this matrix is different from the two matrices obtained previously up to permutation similarity.

Combining the above arguments, we see that there are three different matrices that differ by one switch from the standard matrix up to permutation similarity. $\square$

Note that when $k = 3$, the matrices labeled as A_1, A_2, A_3 in Section 2 are permutationally similar to the three different matrices obtained from the standard matrix A_0 by one switch. The other two matrices A_4 and A_5 can be obtained from A_0 by two switches. The above discussion leads to the following problem:

Problem 3.5 Let $r \geq 2$. Find all matrices that can be obtained by applying r switches to the standard matrix.

Since we need identical rows and identical columns to perform switches, it would also be helpful to resolve the following question:

Problem 3.6 Are there always identical rows and identical columns for matrices in $\mathcal{A}_n$?

4 Sub-Central Groupoids

A proper subset of a central groupoid is called a *sub-central groupoid* if the subset is itself a central groupoid, using the same binary operation. In terms of matrices, if the initial central groupoid is of size k^2, then a sub-central groupoid corresponds to a principal submatrix B of a matrix $A \in \mathcal{A}_{k^2}$ such that $B \in \mathcal{A}_{r^2}$, for some natural number $r < k$. First, we show that every central groupoid of size k^2 can be embedded in a central groupoid of size $(k+1)^2$. Suppose α, β are subsequences of $(1, \ldots, n)$, and C is an $n \times n$ matrix. Denote by $C[\alpha, :]$ the submatrix of C with rows in α, $C[:, \beta]$ the submatrix of C with columns in β, and $C[\alpha, \beta]$ the submatrix of C with rows in α and columns in β.

Theorem 4.1 *Suppose $A \in \mathcal{A}_{k^2}$ for some positive integer k. Let α, β be length k subsequences of $(1, \ldots, k^2)$ such that the sum of rows in $A[\alpha, :]$ equals $\mathbf{1}_{k^2}^t$ and the sum of columns in $A[:, \beta]$ equals $\mathbf{1}_{k^2}$. Then the matrix*

$$B = \begin{pmatrix} A & A[:, \beta] & O_{k^2,1} & O_{k^2,k} \\ O_{k,k^2} & O_{k,k} & \mathbf{1}_k & J_{k,k} \\ O_{1,k^2} & O_{1,k} & 1 & \mathbf{1}_k^t \\ A[\alpha, :] & A[\alpha, \beta] & O_{k,1} & O_{k,k} \end{pmatrix} \in \mathcal{A}_{(k+1)^2}$$

and $\operatorname{rank}(B) = \operatorname{rank}(A) + 1$.

We write B in 4×4 block form so that one can perform block multiplication.

Proof. Since the rows of $A[\alpha, :]$ sum up to $\mathbf{1}_{k^2}^t$, each column of $A[\alpha, :]$ has exactly one nonzero entry. Similarly, each row of $A[:, \beta]$ has exactly one nonzero entry. So, $A[\alpha, \beta]$ is a permutation matrix. Note also that $A \cdot A[\alpha, :]$ and $A[:, \beta] \cdot A$ are matrices with all entries equal to one. With the above observations, one can verify that $B^2 = J_{(k+1)^2}$ by block multiplication.

Finally, observe that

$$\begin{pmatrix} A & A[:, \beta] \\ A[\alpha, :] & A[\alpha, \beta] \end{pmatrix}$$

has the same rank as A, and thus $\operatorname{rank} B = \operatorname{rank} A + 1$. □

We have the following conjecture.

Conjecture 4.2 Let k be a positive integer. Every central groupoid of size k^2 has a sub-central groupoid of size $(k-1)^2$.

We have checked the 101 matrices in $\mathcal{A}_{16}$ in [2]. The conjecture is valid for these matrices. The conjecture also holds for a certain subsets of $\mathcal{A}_n$ for arbitrary $n = k^2$ as shown in [2, Section 6]. Note that if Conjecture 4.2 is true, then every central groupoid of size k^2 can be built from a central groupoid of size $(k-1)^2$. With the help of a computer, we can verify the conjecture for $k = 2, 3$. The general case remains open. In connection to this, we have the following:

Theorem 4.3 *Every central groupoid of size k^2 has at most k different sub-central groupoids of size $(k-1)^2$. This upper bound is attained by the standard central groupoid, i.e., the one whose corresponding matrix is the standard matrix.*

Proof. We prove the result in the context of a central digraph. Let D be a central digraph with k^2 vertices. We need to show that there are at most k different sub-central digraphs in D. By Theorem 2.1, D has k loops, and every sub-central digraph of size $(k-1)^2$ has $k-1$ loops. To prove our assertion, we show that for each loop vertex of D one can get at most one sub-central digraph without that loop vertex. To this end, let $D \setminus \alpha$ and $D \setminus \beta$ be two sub-central digraphs of D of order $(k-1)^2$ obtained from D by deleting vertices in the set

α and the set β, and let α and β contain the same loop vertex. Consider any two vertices u and v in $D \setminus (\alpha \cup \beta)$. There is a walk of length 2 from u to v in $D \setminus \alpha$, and a walk of length 2 from u to v in $D \setminus \beta$. Since D is a central digraph, these walks are the same. So, there is a unique walk between u and v in $D \setminus (\alpha \cup \beta)$. It follows that $D \setminus (\alpha \cup \beta)$ is a central digraph with $(k-1)$ loop vertices. Hence, $D \setminus (\alpha \cup \beta)$ has order $(k-1)^2$. This implies that $\alpha = \beta$. $\square$

Acknowledgment

This research was supported by the NSF REU grant DMS 9987803 at the College of William and Mary. In addition, Li was supported by the NSF grant DMS 0071944. We would like to thank Professor David Stanford for drawing our attention to this problem, and Professor Fletcher for providing preprints. We would also like to thank the referees for many valuable suggestions, including the elegant proof of Theorem 4.3.

References

[1] R. Brualdi and H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, New York, 1991.

[2] F. Curtis and D. Pragel, Central groupoids, central digraphs, and zero-one matrices A satisfying $A^2 = J$, REU report, William and Mary, 2002.

pdf file available at http://www.resnet.wm.edu/˜cklixx/reu02.pdf

[3] R.R. Fletcher III, Unique Path Property Digraphs, Ph.D. dissertation, Emory University, 1991.

[4] R.R. Fletcher III, Using the theory of groups to construct unique path property digraphs, Proceedings of the Thirty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, LA, 2001). Congr. Numer. 153 (2001), 193–209.

[5] A.J. Hoffman, Research Problem 2-11, J. Combinatorial Theory 2 (1967), 393.

[6] D.E. Knuth, Notes on central groupoids, J. Combinatorial Theory 8 (1970), 376-390.

[7] L.E. Shader, On the existence of finite central groupoids of all possible ranks, J. Combinatorial Theory (Series A) 16 (1974), 221-229.

[8] Y.-K. Wu, R.-Z. Li and Q. Li, g-circulant solutions to the $(0,1)$ matrix equation $A^m = J_n$. Linear Algebra Appl. 345 (2002), 195–224.